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SOME REMARKS ON $(k-1)$-CRITICAL SUBGRAPHS OF k-CRITICAL GRAPHS

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A graph G is said to be k-critical if it has chromatic number k, but every proper subgraph of G has a $(k-1)$-coloring. Gallai asked whether every large k-critical graph contains many $(k-1)$-critical subgraphs. We provide some information concerning this question and some related questions.

A graph G is said to be *k-critical* if it has chromatic number k, but every proper subgraph of G is $(k-1)$-colorable. K_2 is the only 2-critical graph and the only 3-critical graphs are the cycles of odd length. For $k \geq 4$, the class of k-critical graphs is quite complicated and no simple characterization of these graphs can be expected. The reader should see Bollobás ([3], Chapter 5), the survey article of Sachs and Stiebitz [8] or the monograph of Toft [11] for an account of some of the literature.

In this article we consider some questions concerning $(k-1)$-critical subgraphs of k-critical graphs. The first of these was raised by Gallai (see [10]), who asked whether every large k-critical graph must contain many $(k-1)$-critical subraphs and, in particular, whether every k-critical graph of order n contains at least n $(k-1)$-critical subgraphs. This is so when $k=3$ and also when $k=n\geq 4$. If G is a k-critical graph, denote by $f_k(G)$ the number of $(k-1)$-critical subgraphs of G and (we abuse notation slightly) let $f_k(n)=\min f_k(G)$, where the minimum is taken over all k-critical graphs of order n. We then have $f_3(n)=n$ if $n\geq 3$ is odd and $f_n(n)=n$ for $n\geq 3$. Gallai's question is then: Is it true that $f_k(n)\geq n$ for $k\geq 4$, $n>k+1$? Toft [12] proved that if e_1 and e_2 are any two distinct edges of a k-critical graph G there is a $(k-1)$-critical subgraph of G containing e_1 but not e_2 and Stiebitz [10] deduced from this result that

$$f_k(n) \geq \log_2 n. \tag{1}$$

One may also derive (1) from the vertex form of the result of Toft, also given in [12].

Our main result concerning Gallai's problem is the following theorem:

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Theorem 1. *For $k \geq 4$, $n > k+1$,*

$$\binom{f_k(n)}{k-1} \geq n. \tag{2}$$

From (2) it is easily seen that the following lower bound holds:

$$f_k(n) > ((k-1)!n)^{\frac{1}{k-1}}. \tag{3}$$

While (3) is stronger than (1), it is far from providing a positive answer to Gallai's question.

It is natural to ask what is the largest possible number $F_k(n)$ of $(k-1)$-critical subgraphs a k-critical graph of order n may have. Of course, $F_3(n) = f_3(n) = n$ for n odd. Theorem 2 gives bounds for $F_4(n)$.

Theorem 2. *For any $c < \frac{1}{2}$ and any $c' > \frac{2}{3e}$*

$$n^{cn} < F_4(n) < (c'n)^n \tag{4}$$

for all sufficiently large n.

Since $F_{k+1}(n+1) \geq F_k(n)+1$, the lower bound for $F_4(n)$ given by (4) is also a lower bound for $F_k(n)$ for each fixed $k \geq 5$ and all sufficiently large n. Our proof of the upper bound in (4) will rely heavily on the fact that the only 3-critical graphs are the odd cycles. We have not been able to obtain any nontrivial upper bound for $F_k(n)$ for $k \geq 5$.

We also consider briefly a question raised by Nešetřil and Rödl (see [14], problem 45): is it true that for $k \geq 4$ every large k-critical graph contains a large $(k-1)$-critical subgraph? For $k=4$, this question was raised by Dirac [5] and a positive answer was given by Kelly and Kelly [7] and by Voss [16], but for $k \geq 5$ the problem is still open. Let $g_k(G)$ denote the order of a largest $(k-1)$-critical subgraph of a k-critical graph G. It follows from the theorem of Toft mentioned in connection with (1) that the following result holds:

Theorem 3. *For any k-critical graph G of order n*

$$f_k(G)g_k(G) \geq n(k-1). \tag{5}$$

We shall give a proof of Theorem 3 that does not depend on the theorem of Toft.

We turn now to the proofs.

Proof of Theorem 1. Let G be a k-critical graph of order n with $t = f_k(n)$ $(k-1)$-critical subgraphs. Let $v \in V(G)$, the vertex set of G. Since G is k-critical, $G-v$ has a $(k-1)$-coloring. Choose such a $(k-1)$-coloring and denote its color classes by V_1^v, V_2^v, …, V_{k-1}^v. For $i = 1, 2, \dots k-1$, let $G^{i,v}$ denote the subgraph of G induced by

$\{v\}\cup\left(\bigcup_{j\neq i} V_j^v\right)$. $G^{i,v}$ is $(k-1)$-chromatic and therefore contains one (at least) of the $(k-1)$-critical subgraphs of G. Choose such a $(k-1)$-critical subgraph and denote it by $H^{i,v}$. There may be several choices; it does not matter which one is picked. Let $S(v)=\{H^{1,v},H^{2,v},\dots H^{k-1,v}\}$. If $u,v\in V(G)$, $u\neq v$, then $S(u)\neq S(v)$, because one of the graphs in $S(v)$ does not contain u, while u is a vertex of every graph in $S(u)$. It follows that $\binom{t}{k-1}\geq n$ and this is (2), so that Theorem 1 is proved. ∎

It will be convenient if we next consider Theorem 3.

Proof of Theorem 3. Let $t = f_k(G)$ and $s = g_k(G)$. Let the vertex set of G be $V = \{v_1,v_2,\dots,v_n\}$ and let $\mathcal{H} = \{H_1,H_2,\dots,H_t\}$ be the set of $(k-1)$-critical subgraphs of G. Form a bipartite graph L whose parts are V and $\mathcal{H}$ and in which v_i is joined to H_j by and edge if v_i is a vertex of H_j. By the argument in the proof of Theorem 1, each v_i has degree at least $k-1$ in L, so that the number of edges of L is at least $n(k-1)$. Also, each H_j has degree at most s, so that the number of edges of L is at most ts. Thus $ts\geq n(k-1)$. ∎

Before giving the proof of Theorem 2 we make some remarks. A graph G is said to be *vertex-k-critical* if for every vertex v of G, $G-v$ is $(k-1)$-colorable. Jensen [in T. R. Jensen and B. Toft, Graph Colouring Problems, Wiley-Interscience, to appear] has proposed the following stronger form of Gallai's conjecture; namely, that the number $l_k(G)$ of vertex-$(k-1)$-critical induced subgraphs of a vertex-k-critical graph G of order n is at least n. An examination of the proof of Theorem 1 shows that the argument given there applies in this context. Thus if $l=l_k(n)=\min l_k(G)$, where the minimum is taken over all vertex-k-critical graphs G of order n, then $\binom{l}{k-1}\geq n$ holds. Also, the proof of Theorem 3 shows that $l_k(G)g_k(G)\geq n(k-1)$.

The referee has suggested that the graph L given in the proof of Theorem 3 may have a matching which covers V. If true, this would give a positive answer to Gallai's question and the analogous conjecture for vertex-k-critical graphs would imply Jensen's conjecture.

We turn now to Theorem 2.

Proof of Theorem 2. We first establish the lower bound. Let m be a positive integer and let V be a set of size $8m+4$. Let $V=V_1\cup V_2\cup V_3\cup V_4$ be a partition of V into parts of size $2m+1$. Let G_m be the graph whose vertex set is V and whose edge set is given as follows:

(i) the edges of a cycle on V_1

(ii) the edges of a cycle on V_4

(iii) the edges of the complete bipartite graph with parts V_2 and V_3

(iv) the edges of a matching from V_1 to V_2

(v) the edges of a matching from V_3 to V_4.

G_m is one of the graphs constructed by Toft [13] in establishing the existence of 4-critical graphs with many edges. It is 4-critical and it is easy to see that it has more than $((2m+1)!)^2$ cycles of odd length. It follows, via Stirling's formula, that the lower bound in (4) holds for infinitely many n. That it holds for all sufficiently large n may be seen by applying Hajós' well known construction [6] to G_m and seven 4-critical graphs of order 10, 11, 12, 13, 14, 15, 16.

It remains to establish the upper bound for $F_4(n)$. Note first, in order to place the result in perspective, that the number of odd cycles in K_n, $n \geq 3$ is $\sum_{1 \leq j \leq [\frac{n-1}{2}]} \binom{n}{2j+1} \frac{(2j)!}{2}$. This sum is clearly an upper bound for $F_4(n)$ and one easily deduces, using Stirling's formula, that $F_4(n) < (c'n)^n$ for any $c' > \frac{1}{e}$ if n is large enough. Our aim is to show that the bound holds for any $c' > \frac{2}{3e}$.

Let G be a 4-critical graph of order n with $F_4(n)$ odd cycles. For $j = 1, 2, \ldots, [\frac{n-1}{2}]$, let there be $\gamma(j)$ odd cycles in G of length $2j+1$ so that

$$\gamma(1) + \gamma(2) + \ldots + \gamma\left(\left[\frac{n-1}{2}\right]\right) = F_4(n).$$

We may choose $l, 1 \leq l \leq [\frac{n-1}{2}]$, so that

$$\gamma(l) \geq \frac{2F_4(n)}{n}. \tag{6}$$

Let $\mathcal{H}$ denote the set of induced subgraphs of G of order $2l+1$ which contain a Hamilton cycle and choose $H \in \mathcal{H}$ so that the number h of Hamilton cycles of H is as large as possible. We then have

$$h \geq \frac{\gamma(l)}{\binom{n}{2l+1}}. \tag{7}$$

Label the vertices of H in some manner and let A be the adjacency matrix of H with respect to this labelling. The number of coverings of the vertex set of H by vertex disjoint cycles (directed and with initial vertex specified) is then the permanent of A. It follows that

$$h \leq \frac{1}{2(2l+1)} \operatorname{Perm}(A). \tag{8}$$

Let $d_1, d_2, \ldots, d_{2l+1}$ be the degree sequence of H. It follows from the theorem of Brégman [4] (see also [9] or [2], pages 23-25) that

$$\operatorname{Perm}(A) \leq \prod_{i=1}^{2l+1} (d_i!)^{1/d_i}. \tag{9}$$

From (6), (7), (8) and (9) it now follows that

$$F_4(n) \le \frac{n\binom{n}{2l+1}}{4(2l+1)} \prod_{i=1}^{2l+1} (d_i!)^{1/d_i}. \tag{10}$$

The product appearing in (10) cannot be larger than that obtained when the degrees of the vertices of H are as nearly equal as possible. See Lemma 2.3 and Corollary 2.3 of the paper of Alon [1]. Thus, if m denotes the number of edges of H and we define u and r by $u=\left[\frac{2m}{2l+1}\right]$ and $ru+(2l+1-r)(u+1)=2m$, we have, from (10),

$$F_4(n) \le \frac{n\binom{n}{2l+1}}{4(2l+1)} (u!)^{\frac{r}{u}} ((u+1)!)^{\frac{2l+1-r}{u+1}}. \tag{11}$$

We also have

$$m \le \frac{(2l+1)^2}{3} \tag{12}$$

since, otherwise, by Turán's Theorem [15], H would contain K_4 and hence could not be a subgraph of a 4-critical graph. It now follows from (11) and (12) and some computations involving Stirling's formula that the upper bound in (4) holds. ∎

Note that the proof actually provides an upper bound for the number of odd cycles contained in a graph not containing K_4.

It would be of interest to know whether there is a constant $c_1 < 1$ for which $F_4(n) < n^{c_1 n}$ for all sufficiently large n.

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References

[1] Noga Alon: The maximum number of Hamiltonian paths in tournaments, *Combinatorica*, **10** (1990), 319–324.

[2] Noga Alon, J. Spencer and P. Erdős, The probabilistic method, *Wiley Interscience Series in Discrete Mathematics and Optimization*, 1992, 23–25.

[3] Béla Bollobás: Extremal Graph Theory, *Academic Press*, 1978.

[4] L. M. Brégman: Some properties of nonnegative matrices and their permanents, *Soviet Math. Dokl.*, **14** (1973), 945–949; [Dokl. Akad. Nauk SSSR, **211**, (1973), 27–30].

[5] G. A. DIRAC: Some theorems on abstract graphs, *Proc. Lond. Math. Soc. Series 3*, **2** (1952), 69–81.

[6] G. HAJÓS: Über eine Konstruktion nicht n-Färbbarer Graphen, *Wiss. Zeit. Martin-Luther Univ. Halle-Wittenberg, Math-Natur.*, **10**, (1961), 116–117.

[7] J. B. KELLY and L. M. KELLY: Paths and circuits in critical graphs, *Amer. Jour. Math.*, **76**, (1954), 786–792.

[8] H. SACHS and M. STIEBITZ: On constructive methods in the theory of colour-critical graphs, *Discrete Mathematics*, **74**, (1989), 201–226.

[9] A. SCHRIJVER: A short proof of Minc's conjecture, *J. Combinatiorial Theory, Series A*, **25**, (1978), 80–83.

[10] M. STIEBITZ: Subgraphs of colour-critical graphs, *Combinatorica*, **7**, (1987), 303–312.

[11] B. TOFT: Graph Colouring Theory, *Odense Universitet*, (1987).

[12] B. TOFT: On critical subgraphs of colour-critical graphs, *Discrete Mathematics*, **7**, (1974), 377–392.

[13] B. TOFT: On the maximal number of edges in critical k-chromatic graphs, *Studia Sci. Math. Hung.*, **5**, (1970), 461–470.

[14] B. TOFT: 75 graph-coloring problems, *"Graph Colouring," Pitman Research Notes in Mathematics Series* (R. Nelson and R. J. Wilson, eds.) vol. **218**, Longman, (1990), 9–35.

[15] P. TURÁN: On an extremal problem in graph theory, *(in Hungarian), Mat. Fiz. Lapok*, **48**, (1941),436–452; *(English translation in: P. Erdős (editor), Collected Papers of Paul Turán*, Volume **1**, Akadémia Kiadó, Budapest, (1990), 231–240.

[16] H. J. VOSS: Graphs with prescribed maximal subgraphs and critical chromatic graphs, *Comment Math. Univ. Carolinae*, **18**, (1977), 129–142.

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